Strong law of large numbers for supercritical superprocesses under second moment condition

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Abstract

Suppose that $X = \{X_t, t \geq 0\}$ is a supercritical superprocess on a locally compact separable metric space (E, m). Suppose that the spatial motion of X is a Hunt process satisfying certain conditions and that the branching mechanism is of the form

$$\psi(x,\lambda) = -a(x)\lambda + b(x)\lambda^2 + \int_{(0,+\infty)} (e^{-\lambda y} - 1 + \lambda y) n(x, dy), \quad x \in E, \quad \lambda > 0,$$

where $a \in \mathcal{B}_b(E)$, $b \in \mathcal{B}_b^+(E)$ and n is a kernel from E to $(0, \infty)$ satisfying

$$\sup_{x \in E} \int_0^\infty y^2 n(x, dy) < \infty.$$

Put $T_t f(x) = \mathbb{P}_{\delta_x}\langle f, X_t\rangle$. Let $\lambda_0 > 0$ be the largest eigenvalue of the generator L of T_t, and ϕ_0 and $\widehat{\phi}_0$ be the eigenfunctions of L and $\widehat{L}$ (the dural of L) respectively associated with λ_0. Under some conditions on the spatial motion and the ϕ_0-transformed semigroup of T_t, we prove that for a large class of suitable functions f, we have

$$\lim_{t\to\infty} e^{-\lambda_0 t}\langle f, X_t\rangle = W_\infty \int_E \widehat{\phi}_0(y) f(y) m(dy), \quad \mathbb{P}_\mu\text{-a.s.},$$

for any finite initial measure μ on E with compact support, where W_∞ is the martingale limit defined by $W_\infty := \lim_{t\to\infty} e^{-\lambda_0 t}\langle \phi_0, X_t\rangle$. Moreover, the exceptional set in the above limit does not depend on the initial measure μ and the function f.

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1 Introduction

Recently there have been quite a few papers on law of large numbers for superdiffusions. In [11, 12, 13] some weak laws of large numbers (convergence in law or in probability) were established. The strong law of large numbers for superprocesses was first studied in [7] followed by [9, 19, 21, 29]. The continuity of the sample paths of the spatial motions played an important role in all the papers mentioned above except [7, 19]. It is more difficult to establish strong law of large numbers for superprocesses with discontinuous spatial motions. For a good survey on recent developments in laws of large numbers for branching Markov processes and superprocesses, see [9]. In the papers mentioned above, either the spatial motion is assumed to be a diffusion, or the spatial motion is assumed to be a symmetric Hunt process. In the paper [7] where the spatial motion is a symmetric Hunt process, a condition on the smallness at "infinity" of the linear term in the branching mechanism of the superprocess has to be assumed. The purpose of this paper is to give a different setup under which the strong law of large number for superprocesses holds. The setup of this paper complements the previous setups. In particular, the spatial motion may be discontinuous and non-symmetric. We will give some examples satisfying the conditions of this paper.

The papers [7, 9, 21] dealt with strong law of large numbers for superprocesses with spatially dependent branching mechansim. The main ideas of the arguments of [7, 9, 21] are similar and consist of two steps. The first step is to prove an almost sure limit result for discrete times, and the second step is to prove that the result is true for continuous times. An essential difficulty comes from the second step. [21] gave a method for the transition from lattice times to continuous times based on the resolvent operator and approximation of the indicator function of an open subset of E by resolvent functions. The reason that this approximation works for superdiffusions is that the sample paths of the spatial motion are continuous. [9] also used this idea to show that indicator functions can be approximated by resolvent functions. For general superprocesses with spatial motions which might be discontinuous, [7] is the first paper to establish a strong law of large numbers under a second moment condition. The paper [7] managed to overcome the difficulty of transition from discrete times to continuous times with a highly non-trivial application of the martingale formulation of superprocesses. However, the assumptions of [7] are restrictive in two aspects: the spatial motion is assumed to be symmetric and the linear term of the branching mechanism is assumed to satisfy a Kato class condition at "infinity".

The papers [29, 19] dealt with strong law of large numbers for super-Brownian motions and super-α-stable processes with spatially independent branching mechanism respectively. The key ingredients in the argument of [29, 19] are Fourier analysis and stochastic analysis, and the conditions in [29, 19] are quite different from those of [7, 21]. The mean semigroup of the superprocess is assumed to have a spectral gap in [7, 21], while the mean semigroups of the superprocesses of [29, 19] have continuous spectra. In this paper we assume that the spatial motion has a dual with respect to a certain measure and that the branching mechanism satisfies a second moment condi-

tion. Under the conditions of this paper, the mean semigroup of the superprocess automatically has a spectral gap.

1.1 Spatial process

Our assumptions on the underlying spatial process are the similar to those in [24]. In this subsection, we recall the assumptions on the spatial process.

Suppose (E, m) is a locally compact separable metric space and m is a σ-finite Borel measure on E with full support. Let $E_\partial = E \cup \{\partial\}$ be the one-point compactification of E. Every function f on E is automatically extended to E_∂ by setting $f(\partial) = 0$. We will assume that $\xi = \{\xi_t, \Pi_x\}$ is a Hunt process on E and $\zeta := \inf\{t > 0 : \xi_t = \partial\}$ is the lifetime of ξ. The transition semigroup of ξ will be denoted by $\{P_t, t \geq 0\}$. We will always assume that there exists a family of strictly positive continuous functions $\{p(t, x, y), t > 0\}$ on $E \times E$ such that

$$P_t f(x) = \int_E p(t, x, y) f(y)\, m(dy).$$

Define

$$a_t(x) := \int_E p(t, x, y)^2\, m(dy), \qquad \widehat{a}_t(x) := \int_E p(t, y, x)^2\, m(dy). \tag{1.1}$$

In this paper, we assume that

Assumption 1.1 **(a)** *For all $t > 0$ and $x \in E$, $\int_E p(t, y, x)\, m(dy) \leq 1$.*

(b) *For any $t > 0$, a_t and $\widehat{a}_t$ are continuous $L^1(E; m)$-integrable functions.*

(c) *There exists $t_0 > 0$ such that $a_{t_0}, \widehat{a}_{t_0} \in L^2(E; m)$.*

By the Chapman-Kolmogorov equation and the Cauchy-Schwarz inequality,

$$p(t + s, x, y) = \int_E p(t, x, z) p(s, z, y)\, m(dz) \leq (a_t(x))^{1/2} (\widehat{a}_s(y))^{1/2}. \tag{1.2}$$

Therefore, $a_{t+s}(x) \leq \int_E \widehat{a}_s(y)\, m(dy) a_t(x)$ and $\widehat{a}_{t+s}(x) \leq \int_E a_s(y)\, m(dy) \widehat{a}_t(x)$. Thus under condition (b), the condition (c) above is equivalent to

(c′) There exists $t_0 > 0$ such that for all $t \geq t_0$, $a_t, \widehat{a}_t \in L^2(E; m)$.

Under Assumption 1.1(a), for every $t > 0$, both P_t and the operator $\widehat{P}_t$ defined by $\widehat{P}_t f(x) = \int_E p(t, y, x) f(y) m(dy)$ are contraction operators in $L^p(E; m)$ for every $p \in [1, \infty]$, and they are dual to each other. Assumption 1.1(b) implies that each P_t is a Hilbert-Schmidt operator in $L^2(E; m)$ and thus is compact. Hence P_t has discrete spectrum.

1.2 Superprocesses

In this subsection, we introduce the superprocesses. Let $\mathcal{B}_b(E)$ (respectively, $\mathcal{B}_b^+(E)$) be the family of bounded (respectively, nonnegative bounded) Borel functions on E. Denote by $\langle\cdot,\cdot\rangle_m$ the inner product in $L^2(E;m)$.

The superprocess $X = \{X_t, t \geq 0\}$ is determined by three parameters: a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E satisfying the assumptions of the previous subsection, a branching rate function $\beta(x)$ on E which is a nonnegative bounded Borel function and a branching mechanism ψ of the form

$$\psi(x,\lambda) = -a(x)\lambda + b(x)\lambda^2 + \int_{(0,+\infty)} (e^{-\lambda y} - 1 + \lambda y)n(x,dy), \quad x \in E, \quad \lambda > 0, \tag{1.3}$$

where $a \in \mathcal{B}_b(E)$, $b \in \mathcal{B}_b^+(E)$ and n is a kernel from E to $(0,\infty)$ satisfying

$$\sup_{x\in E} \int_0^\infty y^2 n(x,dy) < \infty. \tag{1.4}$$

Let $\mathcal{M}_F(E)$ be the space of finite measures on E, equipped with the weak convergence topology. As usual, $\langle f,\mu\rangle := \int f(x)\mu(dx)$ and $\|\mu\| := \langle 1,\mu\rangle$. According to [20, Theorem 5.12], there is a Borel right process $X = \{\Omega, \mathcal{G}, \mathcal{G}_t, X_t, \mathbb{P}_\mu\}$ taking values in $\mathcal{M}_F(E)$, called superprocess, such that for every $f \in \mathcal{B}_b^+(E)$ and $\mu \in \mathcal{M}_F(E)$,

$$-\log \mathbb{P}_\mu \left(e^{-\langle f, X_t\rangle} \right) = \langle u_f(\cdot,t), \mu\rangle, \tag{1.5}$$

where $u_f(x,t)$ is the unique positive solution to the equation

$$u_f(x,t) + \Pi_x \int_0^t \psi(\xi_s, u_f(\xi_s, t-s))\beta(\xi_s) ds = \Pi_x f(\xi_t), \tag{1.6}$$

where $\psi(\partial,\lambda) = 0, \lambda > 0$. Here $(\mathcal{G},\mathcal{G}_t)_{t\geq 0}$ are augmented, $(\mathcal{G}_t, t \geq 0)$ is right continuous and X satisfies the Markov property with respect to $(\mathcal{G}_t, t \geq 0)$. Moreover, such a superprocess X has a Hunt realization in $\mathcal{M}_F(E)$, see [20, Theorem 5.12]. In this paper, the superprocess we deal with always takes such a Hunt realization.

Define

$$\alpha(x) := \beta(x)a(x) \quad \text{and } A(x) := \beta(x)\left(2b(x) + \int_0^\infty y^2 n(x,dy)\right). \tag{1.7}$$

Then, by our assumptions, $\alpha(x) \in \mathcal{B}_b(E)$ and $A(x) \in \mathcal{B}_b^+(E)$. Thus there exists $K > 0$ such that

$$\sup_{x\in E} (|\alpha(x)| + A(x)) \leq K. \tag{1.8}$$

For any $f \in \mathcal{B}_b(E)$ and $(t,x) \in (0,\infty) \times E$, define

$$T_t f(x) := \Pi_x \left[e^{\int_0^t \alpha(\xi_s)\,ds} f(\xi_t) \right]. \tag{1.9}$$

It is well-known that $T_t f(x) = \mathbb{P}_{\delta_x}\langle f, X_t\rangle$ for every $x \in E$. It is known that (see, e.g., [24] and [26, Lemma 2.1]) $\{T_t, t \geq 0\}$ is a strongly continuous semigroup on $L^2(E;m)$ and there exists a function $q(t,x,y)$ on $(0,\infty)\times E\times E$ which is continuous in (x,y) for each $t>0$ such that

$$e^{-Kt}p(t,x,y) \leq q(t,x,y) \leq e^{Kt}p(t,x,y) \quad \text{for } (t,x,y) \in (0,\infty)\times E\times E \tag{1.10}$$

and that for any bounded Borel function f on E and $(t,x) \in (0,\infty)\times E$,

$$T_t f(x) = \int_E q(t,x,y)f(y)m(dy).$$

Define

$$b_t(x) := \int_E q(t,x,y)^2\, m(dy), \qquad \widehat{b}_t(x) := \int_E q(t,y,x)^2\, m(dy). \tag{1.11}$$

Then b_t and $\widehat{b}_t$ enjoy the following properties:

(i) For any $t>0$, we have $b_t, \widehat{b}_t \in L^1(E;m)$. Moreover, $b_t(x)$ and $\widehat{b}_t(x)$ are continuous in $x \in E$.

(ii) There exists $t_0>0$ such that for all $t \geq t_0$, $b_t, \widehat{b}_t \in L^2(E;m)$.

Let $\{\widehat{T}_t, t>0\}$ be the adjoint semigroup of $\{T_t, t\geq 0\}$ on $L^2(E,m)$ defined by

$$\widehat{T}_t g(x) = \int_E q(t,y,x)g(y)\, m(dy).$$

It is easy to see $\widehat{T}_t$ is the dual operator of T_t in $L^2(E;m)$. It follows that $\{\widehat{T}_t, t>0\}$ is also strongly continuous in $L^2(E,m)$. Since $q(t,\cdot,y)$ and a_t are continuous, by (1.2) and (1.10), using the dominated convergence theorem, we get that for any $t>0$ and $f \in L^2(E;m)$, $T_t f$ and $\widehat{T}_t f$ are continuous.

It follows from (i) above that, for any $t>0$, T_t and $\widehat{T}_t$ are compact operators in $L^2(E;m)$. Let L and $\widehat{L}$ be the infinitesimal generators of the semigroups $\{T_t\}$ and $\{\widehat{T}_t\}$ in $L^2(E;m)$ respectively. Let $\sigma(L)$ and $\sigma(\widehat{L})$ be the spectra of L and $\widehat{L}$. It follows from [22, Theorem 2.2.4 and Corollary 2.3.7] that both $\sigma(L)$ and $\sigma(\widehat{L})$ consist of eigenvalues, and that $\sigma(L)$ and $\sigma(\widehat{L})$ have the same number, say N, of eigenvalues. Let $\mathbb{I} = \{0,\cdots,N-1\}$ if $N<\infty$ and $\mathbb{I} = \{0,\cdots\}$ otherwise. Define $\lambda_0 := \sup \Re(\sigma(L)) = \sup \Re(\sigma(\widehat{L}))$. By Jentzsch's theorem (Theorem V.6.6 on page 337 of [27]), λ_0 is an eigenvalue of multiplicity 1 for both L and $\widehat{L}$. Assume that ϕ_0 and $\widehat{\phi}_0$ are the eigenfunctions of L and $\widehat{L}$ respectively associated with λ_0. ϕ_0 and $\widehat{\phi}_0$ can be chosen to be continuous strictly positive and satisfy $\|\phi_0\|_2 = 1$ and $\langle \phi_0, \widehat{\phi}_0\rangle_m = 1$. We list the eigenvalues of $\{\lambda_k, k\in\mathbb{I}\}$ of L in an order so that $\lambda_0 > \Re(\lambda_1) \geq \Re(\lambda_2) \geq \cdots$. Then $\{\overline{\lambda}_k, k \in \mathbb{I}\}$ are the eigenvalues of $\widehat{L}$. For convenience, we define, for any positive integer not in $\mathbb{I}$, $\lambda_k = \overline{\lambda}_k = -\infty$. For $k \in \mathbb{I}$, we write $\Re_k := \Re(\lambda_k)$. We use the convention $\Re_\infty = -\infty$.

For $t>0$, $T_t\phi_0(x) = e^{\lambda_0 t}\phi_0(x)$, and thus

$$\phi_0(x) \leq e^{-\lambda_0 t} b_t(x)^{1/2}. \tag{1.12}$$

Similarly, we have $\widehat{T}_t\widehat{\phi}_0(x) = e^{\lambda_0 t}\widehat{\phi}_0(x)$ and $\widehat{\phi}_0(x) \le e^{-\lambda_0 t}\|\widehat{\phi}_0\|_2\widehat{b}_t(x)^{1/2}$. Therefore, by Assumption 1.1(c), $\phi_0 \in L^2(E;m) \cap L^4(E;m)$. In this paper, we always assume that the superprocess X is supercritical, that is, $\lambda_0 > 0$. Define $W_t := e^{-\lambda_0 t}\langle \phi_0, X_t\rangle$. By the Markov property of X, $\{W_t, t \ge 0\}$ is a nonnegative martingale with respect to $\{\mathcal{G}_t, t \ge 0\}$, and thus the $W_\infty := \lim_{t\to\infty} W_t$ exists. Under our assumptions, W_t is a L^2-bounded martingale, thus W_∞ is non-degenerate, that is $\mathbb{P}_\mu(W_\infty > 0) > 0$.

1.3 Main results

In this subsection, we state our main results. In the remainder of this paper, whenever we talk about an initial configuration $\mu \in \mathcal{M}_F(E)$, we always implicitly assume that it has compact support.

For $q > \max\{K, \lambda_0\}$ and $f \in L^p(E;m)$ with $p \ge 1$, define,

$$U_q f(x) := \begin{cases} \int_0^\infty e^{-qs} T_s f(x)\, ds, & \text{if } \int_0^\infty e^{-qs} T_s |f|(x)\, ds < \infty; \\ 0, & \text{otherwise.} \end{cases}$$

Note that for $p \ge 1$, by Assumption 1.1(a) and (1.10)

$$\begin{aligned} & \left(\int_E \left(\int_0^\infty e^{-qs} T_s |f|(x)\, ds\right)^p m(dx)\right)^{1/p} \\ \le\ & \int_0^\infty e^{-qs} \|T_s(|f|)\|_p\, ds \\ \le\ & \int_0^\infty e^{-qs} e^{Ks}\, ds \|f\|_p < \infty, \end{aligned} \tag{1.13}$$

which implies that $\int_0^\infty e^{-qs} T_s|f|(x)\, ds \in L^p(E;m)$, and thus $\int_0^\infty e^{-qs} T_s|f|(x)\, ds < \infty$, m-a.e. Consequently, $U_q f \in L^p(E;m)$. In Lemma 2.2 below, we will show that if $f \in L^2(E,m) \cap L^4(E,m)$ then $\langle U_q f, X_t\rangle$ is well defined.

Theorem 1.2 *Assume that Assumption 1.1 holds. If $g = U_q f$ for some $f \in L^2(E,m) \cap L^4(E;m)$ and $q > \max\{K, \lambda_0\}$, then for any $\mu \in \mathcal{M}_F(E)$, as $t \to \infty$,*

$$e^{-\lambda_0 t}\langle g, X_t\rangle \to \langle g, \widehat{\phi}_0\rangle_m W_\infty, \quad \mathbb{P}_\mu\text{-a.s.} \tag{1.14}$$

For any $f \ge 0$, define

$$T_t^{\phi_0} f(x) = \frac{e^{-\lambda_0 t}}{\phi_0(x)} \Pi_x \left[\exp\left(\int_0^t \alpha(\xi_s) ds\right) (f\phi_0)(\xi_t)\right]. \tag{1.15}$$

Let $C_0(E;\mathbb{R})$ denote the family of real-valued continuous functions f on E with the property that $\lim_{x\to\partial} f(x) = 0$.

We will also make the following assumption in this paper.

Assumption 1.3 *The semigroup $\{T_t^{\phi_0}, t \geq 0\}$ has the following properties: For any $f \in C_0(E;\mathbb{R})$,*

$$\lim_{t\to 0} \|T_t^{\phi_0} f - f\|_\infty = 0. \tag{1.16}$$

The following theorem is the main result of this paper.

Theorem 1.4 *Under Assumptions 1.1 and 1.3, there exists $\Omega_0 \subset \Omega$ of probability one (that is, $\mathbb{P}_\mu(\Omega_0) = 1$ for every $\mu \in \mathcal{M}_F(E)$) such that, for every $\omega \in \Omega_0$ and for every bounded Borel function f on E satisfying (a) $|f| \leq c\phi_0$ for some $c > 0$ and (b) the set of discontinuous points of f has zero m-measure, we have*

$$\lim_{t\to\infty} e^{-\lambda_0 t}\langle f, X_t\rangle(\omega) = W_\infty(\omega)\int_E \widehat{\phi}_0(y)f(y)m(dy). \tag{1.17}$$

Assumption 1.3 will be used to extend the test functions from resolvent functions $g = U_q f$ with $f \in L^2(E,m) \cap L^4(E;m)$ to functions of the form $g = f\phi_0$ with $f \in C_0(E;\mathbb{R})$. We will give some examples in Section 4 to show that Assumptions 1.1 and 1.3 are satisfied by many interesting superprocesses including super Ornstein-Uhlenbeck processes (both inward and outward) and superprocesses with discontinuous spatial motions.

Remark 1.5 (1) Compared with [7], our spatial motion can be nonsymmetric and we do not assume that $\alpha(x) = \beta(x)a(x)$ is in the Kato class $K_\infty(\xi)$. The latter would require α be in some sense small at ∞ (see [7] for the definition of $K_\infty(\xi)$). In [7], a compact embedding condition (see [7, 2.4]) is also assumed to ensure that the generator of the semigroup $\{T_t, t \geq 0\}$ has a spectral gap. In this paper, we assume instead Assumption 1.1, which implies that the generator of $\{T_t, t \geq 0\}$ has discrete spectrum.

(2) Compared with [21] where the spatial motion is a diffusion, our spatial motion may be discontinuous. The setup of [21] and the setup of the present are also different in the following ways. In [21], the semigroup of the spatial motion is assumed to be intrinsic ultracontractive. This condition is pretty strong and it excludes some interesting examples including the OU process. In this paper, we assume Assumption 1.1 instead, which is weaker than the intrinsic ultracontractive property and is enough to insure that, for resolvent functions g, the limit $\lim_{t\to\infty} e^{-\lambda_0 t}\langle g, X_t\rangle$ exists almost surely. In [21], the branching mechanism is assumed to satisfy a $L\log L$ condition, while in this paper, we assume that the branching mechanism satisfies a second moment condition.

2 Preliminaries

2.1 Moment estimates

By [24, Lemma 2.2] with $k = 1$, for any $t_1 > 0$ and $a < -\Re(\lambda_1)$, there exists a constant $c = c(a, t_1) > 0$ such that for all $(t, x, y) \in (2t_1, \infty) \times E \times E$,

$$\left|q(t,x,y) - e^{\lambda_0 t}\phi_0(x)\widehat{\phi}_0(y)\right| \leq ce^{-at} b_{t_1}(x)^{1/2}\widehat{b}_{t_1}(y)^{1/2}. \tag{2.1}$$

Multiplying both sides by $e^{-\lambda_0 t}$, we get that for all $(t,x,y)\in(2t_1,\infty)\times E\times E$,

$$\left|e^{-\lambda_0 t}q(t,x,y)-\phi_0(x)\widehat{\phi}_0(y)\right|\le ce^{-(a+\lambda_0)t}b_{t_1}(x)^{1/2}\widehat{b}_{t_1}(y)^{1/2}.$$

Note that $a<-\Re(\lambda_1)$ is equivalent to $a+\lambda_0<\lambda_0-\Re(\lambda_1)$. Thus for any $\widetilde{a}\in(0,\lambda_0-\Re(\lambda_1))$ and $t_1>0$, there exists $c_1=c_1(\widetilde{a},t_1)>0$ such that for all $(t,x,y)\in(2t_1,\infty)\times E\times E$,

$$\left|e^{-\lambda_0 t}q(t,x,y)-\phi_0(x)\widehat{\phi}_0(y)\right|\le c_1e^{-\widetilde{a}t}b_{t_1}(x)^{1/2}\widehat{b}_{t_1}(y)^{1/2}. \tag{2.2}$$

Thus, for $f\in L^2(E;m)$, we have for all $(t,x)\in(2t_1,\infty)\times E$,

$$\left|e^{-\lambda_0 t}T_tf(x)-\phi_0(x)\langle f,\widehat{\phi}_0\rangle_m\right|\le c_1\|\widehat{b}_{t_1}^{1/2}\|_2\|f\|_2e^{-\widetilde{a}t}b_{t_1}(x)^{1/2},$$

which implies that there exists $c_2=c_2(\widetilde{a},t_1)>0$ such that for all $(t,x)\in(2t_1,\infty)\times E$,

$$\left|e^{-\lambda_0 t}T_tf(x)-\phi_0(x)\langle f,\widehat{\phi}_0\rangle_m\right|\le c_2\|f\|_2e^{-\widetilde{a}t}b_{t_1}(x)^{1/2}. \tag{2.3}$$

Hence, by (1.12), we have

$$\begin{aligned}e^{-\lambda_0 t}|T_tf(x)| &\le \phi_0(x)|\langle f,\widehat{\phi}_0\rangle_m|+c_2\|f\|_2e^{-\widetilde{a}t}b_{t_1}(x)^{1/2}\\ &\le (e^{-\lambda_0 t_1}\|\widehat{\phi}_0\|_2+c_2)\|f\|_2b_{t_1}(x)^{1/2}.\end{aligned}$$

Thus there exists $c_3=c_3(\widetilde{a},t_1)>0$ such that for all $(t,x)\in(2t_1,\infty)\times E$,

$$|T_tf(x)|\le c_3\|f\|_2e^{\lambda_0 t}b_{t_1}(x)^{1/2}. \tag{2.4}$$

We now recall the second moment formula for the superprocess $\{X_t,t\ge 0\}$ (see, for example, [23]): for $f\in L^2(E;m)\cap L^4(E;m)$ and $\mu\in\mathcal{M}_F(E)$, we have for any $t>0$,

$$\mathbb{V}\mathrm{ar}_\mu\langle f,X_t\rangle=\langle\mathbb{V}\mathrm{ar}_{\delta_\cdot}\langle f,X_t\rangle,\mu\rangle=\int_E\int_0^t T_s[A(T_{t-s}f)^2](x)\,ds\mu(dx), \tag{2.5}$$

where $\mathbb{V}\mathrm{ar}_\mu$ stands for the variance under $\mathbb{P}_\mu$ and $A(x)$ is the function defined in (1.7). Moreover, for $f\in L^2(E;m)\cap L^4(E;m)$,

$$\mathbb{V}\mathrm{ar}_{\delta_x}\langle f,X_t\rangle\le e^{Kt}T_t(f^2)(x)\in L^2(E;m). \tag{2.6}$$

In the following lemma, we give a useful estimate on the second moment of X. If we choose the constant $\widetilde{a}\in(0,\lambda_0-\Re(\lambda_1))$ small enough, we can get the next lemma by [24, Lemma 2.5]. Here we give a direct proof.

Lemma 2.1 *Suppose that Assumption 1.1 holds. For any $\widetilde{a}\in(0,(\lambda_0-\Re(\lambda_1))\wedge(\lambda_0/2))$ and $f\in L^2(E;m)\cap L^4(E;m)$ with $\langle f,\widehat{\phi}_0\rangle_m=0$, there exists $c_4=c_4(t_0,\widetilde{a},f)>0$ such that*

$$\sup_{t>10t_0}e^{2(-\lambda_0+\widetilde{a})t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f,X_t\rangle\le c_4b_{t_0}(x)^{1/2}. \tag{2.7}$$

Proof: In the following proof, we use $c = c(t_0, \tilde{a}, f)$ to denote a constant whose value may change from one appearance to another. Recall that

$$\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle = \Big(\int_0^{2t_0} + \int_{2t_0}^{t-2t_0} + \int_{t-2t_0}^{t}\Big) T_s[A(T_{t-s}f)^2](x)\, ds.$$

In the following we will deal with the above three parts separately.

(i) For $t > 10t_0$ and $s < 2t_0$, by (2.3), we have

$$|T_{t-s}f(x)| \le ce^{(\lambda_0-\widetilde{a})(t-s)} b_{4t_0}(x)^{1/2}.$$

Thus,

$$\int_0^{2t_0} T_s[A(T_{t-s}f)^2](x)\, ds \le ce^{2(\lambda_0-\widetilde{a})t} \int_0^{2t_0} T_s[b_{4t_0}](x)\, ds.$$

If we can prove that

$$\int_0^{2t_0} T_s[b_{4t_0}](x)\, ds \le cb_{t_0}(x)^{1/2}, \tag{2.8}$$

we will get

$$\int_0^{2t_0} T_s[A(T_{t-s}f)^2](x)\, ds \le ce^{2(\lambda_0-\widetilde{a})t} b_{t_0}(x)^{1/2}. \tag{2.9}$$

Now we prove (2.8). By Fubini's theorem and Hölder's inequality, we get

$$\begin{aligned} a_{t+s}(x) &= \int_E p(t+s,x,y) \int_E p(t,x,z)p(s,z,y)\, m(dz)\, m(dy) \\ &= \int_E p(t,x,z) \int_E p(t+s,x,y)p(s,z,y)\, m(dy)\, m(dz) \\ &\le a_{t+s}(x)^{1/2} \int_E p(t,x,z)a_s(z)^{1/2}\, m(dz) \end{aligned}$$

which implies

$$a_{t+s}(x) \le \left(\int_E p(t,x,z)a_s(z)^{1/2}\, m(dz)\right)^2 \le \int_E p(t,x,z)a_s(z)\, m(dz). \tag{2.10}$$

By (2.10), we get

$$b_{4t_0}(x) \le e^{8Kt_0} a_{4t_0}(x) \le e^{10Kt_0} T_{2t_0}(a_{2t_0})(x).$$

Thus, by Assumption 1.1(c$'$) and (2.4), we have

$$\begin{aligned} \int_0^{2t_0} T_s(b_{4t_0})(x)\, ds &\le e^{10Kt_0} \int_0^{2t_0} T_{s+2t_0}(a_{2t_0})(x)\, ds \\ &\le c\int_0^{2t_0} e^{\lambda_0(s+2t_0)}\, ds b_{t_0}(x)^{1/2} \le cb_{t_0}(x)^{1/2}. \end{aligned} \tag{2.11}$$

Therefore (2.8) holds.

(ii) For $t > 10t_0$ and $s \in (2t_0, t-2t_0)$, by (2.3), (2.4) and Assumption 1.1(c′),

$$T_s[A(T_{t-s}f)^2](x) \le ce^{2(\lambda_0-\widetilde{a})(t-s)}T_s(b_{t_0})(x) \le ce^{2(\lambda_0-\widetilde{a})(t-s)}e^{\lambda_0 s}b_{t_0}(x)^{1/2}.$$

Thus, using the fact $\lambda_0 - 2\widetilde{a} > 0$,

$$\begin{aligned}\int_{2t_0}^{t-2t_0} T_s[A(T_{t-s}f)^2](x)\,ds &\le ce^{2(\lambda_0-\widetilde{a})t}\int_{2t_0}^{t-2t_0} e^{-(\lambda_0-2\widetilde{a})s}\,ds b_{t_0}(x)^{1/2}\\ &\le ce^{2(\lambda_0-\widetilde{a})t}b_{t_0}(x)^{1/2}.\end{aligned}\tag{2.12}$$

(iii) For $t > 10t_0$ and $s > t-2t_0$, since $|T_{t-s}f(x)|^2 \le e^{K(t-s)}T_{t-s}(f^2)(x)$,

$$\begin{aligned}T_s[A(T_{t-s}f)^2](x) &\le Ke^{K(t-s)}T_t(f^2)(x) \le Ke^{2t_0K}c_3e^{\lambda_0 t}b_{t_0}(x)^{1/2}\\ &\le Ke^{2t_0K}c_3e^{2(\lambda_0-\widetilde{a})t}b_{t_0}(x)^{1/2},\end{aligned}$$

where in the last equality we use the fact $\lambda_0 - 2\widetilde{a} > 0$. Thus,

$$\int_{t-2t_0}^{t} T_s[A(T_{t-s}f)^2](x)\,ds \le ce^{2(\lambda_0-\widetilde{a})t}b_{t_0}(x)^{1/2}.\tag{2.13}$$

Combining (2.9), (2.12) and (2.13), we get (2.7). □

2.2 Martingale measure for superprocesses

In this subsection, we recall the associated martingale measure for the superprocess X. For more details, see, for instance, [20, Chapter 7]. The martingale measure for superprocesses is a very useful tool in the proof of our main theorems.

For our superprocess X, there exists a worthy $(\mathcal{G}_t)$-martingale measure $\{M_t(B) = M(t,B); t \ge 0, B \in \mathcal{B}(E)\}$ with covariation measure

$$\nu(ds,dx,dy) := ds\int_E A(z)\delta_z(dx)\delta_z(dy)X_s(dz)$$

such that for $t \ge 0$ and $f \in L^2(E;m) \cap L^4(E;m)$, we have, $\mathbb{P}_\mu$-a.s.,

$$\langle f, X_t\rangle = \langle T_t f, \mu\rangle + \int_0^t\int_E T_{t-s}f(z)\,M(ds,dz).\tag{2.14}$$

For any $u > 0$ and $0 \le t \le u$, we define

$$M_t^{(u)} := \int_0^t\int_E T_{u-s}f(x)M(ds,dx).$$

Then, for any $\mu \in \mathcal{M}_F(E)$, $\{M_t^{(u)}, 0 \le t \le u\}$ is a cadlag square-integrable martingale under $\mathbb{P}_\mu$ with

$$\langle M^u\rangle_t = \int_0^t \langle A(T_{u-s}f)^2, X_s\rangle\,ds.\tag{2.15}$$

Here cadlag means "right continuous having left limits". Note that

$$\mathbb{P}_\mu(M_u^{(u)})^2 = \mathbb{P}_\mu\langle M^u\rangle_u = \mathbb{V}ar_\mu\langle f, X_u\rangle. \tag{2.16}$$

In the remainder of this paper, we will always assume that $q > \max\{K, \lambda_0\}$.

Lemma 2.2 *Assume that Assumption 1.1 holds. If $f \in L^2(E;m) \cap L^4(E;m)$, then for any $\mu \in \mathcal{M}_F(E)$,*

$$\mathbb{P}_\mu\left(\langle U_q|f|, X_t\rangle < \infty \text{ for } t \geq 0\right) = \mathbb{P}_\mu\left(\langle U_q f, X_t\rangle \text{ is finite for } t \geq 0\right) = 1.$$

Moreover, $\mathbb{P}_\mu$-a.s., $\langle U_q f, X_t\rangle$ is cadlag on $[0,\infty)$, and for all $t > 0$,

$$\langle U_q f, X_t\rangle = \langle T_t(U_q f), \mu\rangle + e^{qt}\int_t^\infty e^{-qu} M_t^{(u)}\, du. \tag{2.17}$$

Proof: When the spatial motion ξ is symmetric, this lemma has been established in [25, lemma 2.4 and Lemma 2.5]. The proof for the non-symmetric case is almost the same. For reader's convenience, we include a proof here. We can check that the argument in the proof of [25, Lemma 2.4] works without the assumption that ξ is m-symmetric, so $\langle U_q f, X_t\rangle$ is right continuous on $[0,\infty)$, $\mathbb{P}_\mu$-a.s.

For $f \in L^2(E;m)\cap L^4(E;m)$, $U_q f \in L^2(E;m)\cap L^4(E;m)$. By (2.14), for $t > 0$ and $\mu \in \mathcal{M}_F(E)$, we have, $\mathbb{P}_\mu$-a.s.,

$$\begin{aligned}
\langle U_q f, X_t\rangle &= \langle T_t(U_q f), \mu\rangle + \int_0^t \int_E T_{t-s}(U_q f)(z) M(ds, dz) \\
&= \langle T_t(U_q f), \mu\rangle + \int_0^t \int_E \int_0^\infty e^{-qu} T_{u+t-s} f(z)\, du M(ds, dz) \\
&= \langle T_t(U_q f), \mu\rangle + e^{qt}\int_0^t \int_E \int_t^\infty e^{-qu} T_{u-s} f(z)\, du M(ds, dz) \\
&= \langle T_t(U_q f), \mu\rangle + e^{qt}\int_t^\infty e^{-qu}\, du \int_0^t \int_E T_{u-s} f(z) M(ds, dz) \\
&:= J_1^f(t) + e^{qt} J_2^f(t),
\end{aligned} \tag{2.18}$$

where the fourth equality follows from the stochastic Fubini's theorem for martingale measures (see, for instance, [20, Theorem 7.24]). Thus, for $t > 0$ and $\mu \in \mathcal{M}_F(E)$,

$$\mathbb{P}_\mu\left(\langle U_q f, X_t\rangle = J_1^f(t) + e^{qt} J_2^f(t)\right) = 1. \tag{2.19}$$

Then, in light of (2.19), to prove (2.17), it suffices to prove that $J_1^f(t)$ and $J_2^f(t)$ are all cadlag in $(0,\infty)$, $\mathbb{P}_\mu$-a.s. For $J_1^f(t)$, by Fubini's theorem, for $t > 0$,

$$J_1^f(t) = e^{qt}\int_t^\infty e^{-qs}\langle T_s f, \mu\rangle\, ds.$$

Thus, it is easy to see that $J_1^f(t)$ is continuous in $t \in (0, \infty)$. Now, we consider $J_2^f(t)$. We claim that, for any $t_1 > 0$,

$$\mathbb{P}_\mu \left(J_2^f(t) \text{ is cadlag in } [t_1, \infty) \right) = 1. \tag{2.20}$$

By the definition of J_2^f, for $t \geq t_1$,

$$J_2^f(t) = \int_{t_1}^\infty e^{-qu} M_t^{(u)} 1_{t<u}\, du.$$

Since $t \mapsto M_t^{(u)} \mathbf{1}_{t<u}$ is right continuous, by the dominated convergence theorem, to prove (2.20), it suffices to show that

$$\mathbb{P}_\mu \left(\int_{t_1}^\infty e^{-qu} \sup_{t \geq t_1} \left(|M_t^{(u)}| 1_{t<u} \right) du < \infty \right) = 1. \tag{2.21}$$

By the L^p-maximum inequality and (2.16), we have

$$\begin{aligned}
\mathbb{P}_\mu \left(\int_{t_1}^\infty e^{-qu} \sup_{t \geq t_1} \left(|M_t^{(u)}| 1_{t<u} \right) du \right) &\leq 2 \int_{t_1}^\infty e^{-qu} \sqrt{\mathbb{P}_\mu \left| M_u^{(u)} \right|^2}\, du \\
= \quad 2 \int_{t_1}^\infty e^{-qu} \sqrt{\int_E \mathbb{V}ar_{\delta_x} \langle f, X_u \rangle\, \mu(dx)}\, du.
\end{aligned} \tag{2.22}$$

By (2.6) and (2.4), we have, for $u > t_1$,

$$\begin{aligned}
\int_E \mathbb{V}ar_{\delta_x} \langle f, X_u \rangle\, \mu(dx) &\leq e^{Ku} \int_E T_u(f^2)(x)\, \mu(dx) \\
&\leq c e^{Ku} e^{\lambda_0 u} \int_E b_{t_1/2}(x)^{1/2}\, \mu(dx),
\end{aligned}$$

where $c = c(t_1, \tilde{a}, f)$ is a positive constant and $b_t(x)$ is the function defined in (1.11). Since $x \mapsto b_{t_1/2}(x)$ is continuous and μ has compact support, we have $\int_E b_{t_1/2}(x)^{1/2} \mu(dx) < \infty$. Thus by (2.22), we have

$$\begin{aligned}
&\mathbb{P}_\mu \left(\int_{t_1}^\infty e^{-qu} \sup_{t \geq t_1} \left(|M_t^{(u)}| 1_{t<u} \right) du \right) \\
\leq \quad & 2\sqrt{c} \int_{t_1}^\infty e^{-qu} e^{(K+\lambda_0)u/2}\, du \sqrt{\int_E b_{t_1/2}(x)^{1/2}\, \mu(dx)} < \infty.
\end{aligned}$$

Now (2.21) follows immediately. Since $t_1 > 0$ is arbitrary, we have

$$\mathbb{P}_\mu \left(J_2^f(t) \text{ is cadlag in } (0, \infty) \right) = 1.$$

The proof is now complete. $\square$

3 Strong law of large numbers

In this section, we give the proofs of Theorems 1.2 and 1.4. We start with a lemma.

Lemma 3.1 *Suppose that Assumption 1.1 holds and $f \in L^2(E;m) \cap L^4(E;m)$ with $\langle f, \widehat{\phi}_0\rangle_m = 0$. Then for any $\mu \in \mathcal{M}_F(E)$ and $\widetilde{a} \in (0, (\lambda_0 - \Re(\lambda_1)) \wedge (\lambda_0/2))$,*

$$\sup_{n>10t_0} e^{(-\lambda_0+\widetilde{a})n}\mathbb{P}_\mu\left(\sup_{n\le t\le n+1} |\langle U_q f, X_t\rangle|\right) < \infty. \tag{3.1}$$

Proof: In this proof, we always assume that $n > 10t_0$ and c is a positive constant whose value does not depend on n and may change from one appearance to another. Define $J_1^f(t) := \langle T_t U_q f, \mu\rangle$ and $J_2^f(t) := \int_t^\infty e^{-qu} M_t^{(u)}\, du$. By (2.18), for any $t > 0$,

$$\mathbb{P}_\mu\left(\sup_{n\le t\le n+1} |\langle U_q f, X_t\rangle|\right) \le \sup_{n\le t\le n+1} |J_1^f(t)| + e^{q(n+1)}\mathbb{P}_\mu\left(\sup_{n\le t\le n+1} |J_2^f(t)|\right).$$

First we consider $J_1^f(t)$. Since $\langle U_q f, \widehat{\phi}_0\rangle = 0$, by (2.3), we have $|T_t U_q f|(x) \le c e^{(\lambda_0-\widetilde{a})t} b_{t_0}(x)^{1/2}$. Thus for $n > 10t_0$

$$\begin{aligned}
\sup_{n\le t\le n+1} |J_1^f(t)| &\le \sup_{n\le t\le n+1} \langle |T_t U_q f|, \mu\rangle \\
&\le c \sup_{n\le t\le n+1} e^{(\lambda_0-\widetilde{a})t}\langle b_{t_0}^{1/2}, \mu\rangle \\
&\le c e^{(\lambda_0-\widetilde{a})n}.
\end{aligned} \tag{3.2}$$

Next we deal with $J_2^f(t)$. For $t \in [n, n+1]$,

$$J_2^f(t) = \int_t^\infty e^{-qu} M_t^{(u)}\, du = \int_n^\infty e^{-qu} M_t^{(u)} \mathbf{1}_{t<u}\, du.$$

Thus for $n > 10t_0$,

$$\begin{aligned}
&\mathbb{P}_\mu\left(\sup_{n\le t\le n+1} |J_2^f(t)|\right) \le \int_n^\infty e^{-qu}\mathbb{P}_\mu\Big(\sup_{n\le t\le n+1}\left(\left|M_t^{(u)}\right| 1_{t<u}\right)\Big)\, du \\
\le\ & 2\int_n^\infty e^{-qu}\sqrt{\mathbb{P}_\mu(M_u^{(u)})^2}\, du \le 2\sqrt{c\langle b_{t_0}^{1/2}, \mu\rangle}\int_n^\infty e^{-qu} e^{(\lambda_0-\widetilde{a})u}\, du \\
\le\ & c(q - \lambda_0 + \widetilde{a})^{-1} e^{-(q-\lambda_0+\widetilde{a})n},
\end{aligned}$$

where the third equality follows from (2.15), (2.16) and (2.7). It follows that for $n > 10t_0$,

$$e^{q(n+1)}\mathbb{P}_\mu\left(\sup_{n\le t\le n+1} J_2^f(t)\right) \le c e^{(\lambda_0-\widetilde{a})n}. \tag{3.3}$$

Combining (3.2) and (3.3), this yields (3.1). The proof is now complete. □

Proof of Theorem 1.2: Put $\tilde{f} = f - \langle f, \widehat{\phi}_0\rangle_m \phi_0$. Note that

$$U_q\phi_0(x) = \int_0^\infty e^{-qt} T_t\phi_0(x)\, dt = \int_0^\infty e^{-qt} e^{\lambda_0 t}\, dt \phi_0(x) = (q-\lambda_0)^{-1}\phi_0(x)$$

and

$$\begin{aligned} \langle U_q f, \widehat{\phi}_0\rangle_m &= \int_0^\infty e^{-qt}\langle T_t f, \widehat{\phi}_0\rangle_m\, dt \\ &= \int_0^\infty e^{-qt} e^{\lambda_0 t}\, dt \langle f, \widehat{\phi}_0\rangle_m = (q-\lambda_0)^{-1}\langle f, \widehat{\phi}_0\rangle_m. \end{aligned} \tag{3.4}$$

Thus,

$$U_q f(x) = \langle f, \widehat{\phi}_0\rangle_m U_q\phi_0(x) + U_q(\tilde{f})(x) = \langle U_q f, \widehat{\phi}_0\rangle_m \phi_0(x) + U_q(\tilde{f})(x).$$

Hence, to prove (1.14), we only need to show that

$$e^{-\lambda_0 t}\langle U_q(\tilde{f}), X_t\rangle \to 0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.5}$$

Let $M_n := \sup_{n\le t\le n+1} e^{-\lambda_0 t}\left|\langle U_q(\tilde{f}), X_t\rangle\right|$. By (3.1), there is a constant $c > 0$ so that $\mathbb{P}_\mu M_n \le ce^{-\widetilde{a}n}$ for every $n > 10t_0$. We conclude by the Borel-Cantelli lemma that $M_n \to 0$, as $n \to \infty$, $\mathbb{P}_\mu$-a.s., from which (3.5) follows immediately. The proof is now complete. □

For any $f \ge 0$ and $q > \max\{K, \lambda_0\}$, define

$$U_q^{\phi_0} f(x) = \int_0^\infty e^{-qt} T_t^{\phi_0} f(x) dt, \quad x \in E,$$

where $T_t^{\phi_0}$ is defined in (1.15). It is easy to see that $\phi_0(x) U_q^{\phi_0} f(x) = U_{q+\lambda_0}(\phi_0 f)$.

Proposition 3.2 *Suppose that Assumptions 1.1 and 1.3 hold. For any $0 \le f \in C_0(E;\mathbb{R})$ and $\mu \in \mathcal{M}_F(E)$,*

$$\lim_{t\to\infty} e^{-\lambda_0 t}\langle \phi_0 f, X_t\rangle = \langle f\phi_0, \widehat{\phi}_0\rangle_m W_\infty, \quad \mathbb{P}_\mu\text{-}a.s. \tag{3.6}$$

Proof: By Theorem 1.2,

$$\begin{aligned} \lim_{t\to\infty} e^{-\lambda_0 t}\langle \phi_0 U_q^{\phi_0} f, X_t\rangle &= \lim_{t\to\infty} e^{-\lambda_0 t}\langle U_{q+\lambda_0}(\phi_0 f), X_t\rangle \\ &= \langle U_{q+\lambda_0}(\phi_0 f), \widehat{\phi}_0\rangle_m W_\infty, \quad \mathbb{P}_\mu\text{-a.s.} \end{aligned}$$

According to (3.4),

$$\langle U_{q+\lambda_0}(\phi_0 f), \widehat{\phi}_0\rangle_m = \frac{1}{q}\langle \phi_0 f, \widehat{\phi}_0\rangle_m.$$

Therefore, for any $q > \max\{K, \lambda_0\}$,

$$\lim_{t\to\infty} e^{-\lambda_0 t}\langle \phi_0 q U_q^{\phi_0} f, X_t\rangle = \langle f\phi_0, \widehat{\phi}_0\rangle_m W_\infty, \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.7}$$

Choose a sequence $q_k > \max\{K, \lambda_0\}$ so that $\lim_{k\to\infty} q_k = \infty$. Put

$$\begin{aligned}\Omega^* : \;&= \bigcap_{k\ge 1}\Big\{\lim_{t\to\infty} e^{-\lambda_0 t}\langle \phi_0 q_k U^{\phi_0}_{q_k} f, X_t(\omega)\rangle = \langle f\phi_0, \widehat{\phi}_0\rangle_m W_\infty(\omega)\Big\}\\ &\bigcap\Big\{\lim_{t\to\infty} W_t(\omega) = W_\infty(\omega)\Big\}.\end{aligned}$$

Then $\mathbb{P}_\mu(\Omega^*) = 1$. Note that, for any $\omega \in \Omega^*$,

$$\begin{aligned}&\Big|e^{-\lambda_0 t}\langle \phi_0 q_k U^{\phi_0}_{q_k} f, X_t(\omega)\rangle - e^{-\lambda_0 t}\langle \phi_0 f, X_t(\omega)\rangle\Big|\\ \le\;& e^{-\lambda_0 t}\langle \phi_0 |q_k U^{\phi_0}_{q_k} f - f|, X_t(\omega)\rangle\\ \le\;& \|q_k U^{\phi_0}_{q_k} f - f\|_\infty e^{-\lambda_0 t}\langle \phi_0, X_t(\omega)\rangle,\end{aligned}$$

where $\|\cdot\|_\infty$ is the L^∞ norm. Letting $t\to\infty$, we obtain that,

$$\limsup_{t\to\infty}\Big|e^{-\lambda_0 t}\langle \phi_0 q_k U^{\phi_0}_{q_k} f, X_t(\omega)\rangle - e^{-\lambda_0 t}\langle \phi_0 f, X_t\rangle\Big| \le \|q_k U^{\phi_0}_{q_k} f - f\|_\infty W_\infty(\omega). \tag{3.8}$$

By Assumption 1.3, $\lim_{k\to\infty}\|q_k U^{\phi_0}_{q_k} f - f\|_\infty = 0$. Thus (3.8) implies that, for $\omega\in\Omega^*$,

$$\lim_{k\to\infty}\limsup_{t\to\infty}\Big|e^{-\lambda_0 t}\langle \phi_0 q_k U^{\phi_0}_{q_k} f, X_t(\omega)\rangle - e^{-\lambda_0 t}\langle \phi_0 f, X_t(\omega)\rangle\Big| = 0. \tag{3.9}$$

Now, combining (3.7) and (3.9), we get (3.6). □

Proof of Theorem 1.4: Note that E_∂ is a compact separable metric space. According to [28, Exercise 9.1.16(iii)], $C_b(E_\partial;\mathbb{R})$, the space of bounded continuous $\mathbb{R}$-valued functions f on E, is separable. Therefore $C_0(E;\mathbb{R})$ is also a separable space. Let $\{f_n, n\ge 1\}$ be a countable dense subset of $C_0(E;\mathbb{R})$. Define

$$\begin{aligned}\Omega_0 \;:=\; &\bigcap_{k\ge 1}\left\{\omega\in\Omega : \lim_{t\to\infty} e^{-\lambda_0 t}\langle f_k\phi_0, X_t\rangle(\omega) = W_\infty(\omega)\int_E f_k(y)\phi_0(y)\widehat{\phi}_0(y)m(dy)\right\}\\ &\bigcap\left\{\omega\in\Omega : \lim_{t\to\infty} W_t(\omega) = W_\infty(\omega)\right\}.\end{aligned}$$

By Proposition 3.2, $\mathbb{P}_\mu(\Omega_0) = 1$ for any $\mu\in\mathcal{M}_F(E)$.

We first consider (1.17) on $\{W_\infty > 0\}$. For each $\omega\in\Omega_0\cap\{W_\infty>0\}$ and $t\ge 0$, we define two probability measures ν_t and ν on D, respectively by

$$\nu_t(F)(\omega) = \frac{e^{-\lambda_0 t}\langle 1_F\phi_0, X_t\rangle(\omega)}{W_t(\omega)},\ \text{ and } \nu(F) = \int_F \phi_0(y)\widehat{\phi}_0(y)m(dy),\quad F\in\mathcal{B}(E).$$

Note that the measure ν_t is well-defined for every $t\ge 0$, and ν_t and ν are probability measures. By the definition of Ω_0 we know that ν_t converges weakly to ν as $t\to\infty$. Since ϕ_0 is strictly positive and continuous on E, if f is a function on E such that $|f|\le c\phi_0$ for some $c>0$ and that

the discontinuity set of f has zero m-measure (equivalently zero ν-measure), then $g := f/\phi$ is a bounded function with the same set of discontinuity. We thus have

$$\lim_{t\to\infty}\int_E g(x)\nu_t(\mathrm{d}x) = \int_E g(x)\nu(\mathrm{d}x),$$

which is equivalent to

$$\lim_{t\to\infty} e^{-\lambda_0 t}\langle f, X_t\rangle(\omega) = W_\infty(\omega)\int_E \widehat{\phi}_0(y)f(y)m(\mathrm{d}y) \quad \text{for } \omega\in\Omega_0\cap\{M_\infty(\phi)>0\}.$$

If $|f|\le c\phi_0$ for some positive constant $c>0$, (1.17) holds automatically on $\{W_\infty = 0\}$. This completes the proof of the theorem. □

4 Examples

In this section we give some examples. The main purpose is to illustrate the diverse situations where the main result of this paper can be applied. We will not try to give the most general examples possible.

Example 4.1 (Super inward Ornstein-Uhlenbeck processes) Let $d\ge 1$, $E=\mathbb{R}^d$. Suppose the spatial motion $\xi=\{\xi_t,\Pi_x\}$ is an Ornstein-Uhlenbeck (OU) process on $\mathbb{R}^d$ with infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\sigma^2\Delta - cx\cdot\nabla \text{ on } \mathbb{R}^d,$$

where $\sigma,\ c>0$. Without loss of generality, we assume $\sigma=1$. Let $\varphi(x) := (c/\pi)^{d/2}\, e^{-c\|x\|^2}$, and $m(dx)=\varphi(x)dx$. Then ξ is symmetric with respect to the probability measure $m(dx)$. Suppose that the branching rate function $\beta(x)=\beta$ is a positive constant, and the branching mechanism ψ is given by

$$\psi(x,\lambda) = -\lambda + b(x)\lambda^2 + \int_{(0,+\infty)}(e^{-\lambda y}-1+\lambda y)n(x,dy), \quad x\in\mathbb{R}^d,\ \lambda>0, \tag{4.1}$$

where $b\in\mathcal{B}_b^+(\mathbb{R}^d)$ and n is a kernel from $\mathbb{R}^d$ to $(0,\infty)$ satisfying

$$\sup_{x\in\mathbb{R}^d}\int_0^\infty y^2 n(x,dy)<\infty.$$

Then for the corresponding superprocess,

$$T_t f(x) = e^{\beta t}\Pi_x\left[f(\xi_t)\right] = e^{\beta t}P_t f(x).$$

It is easy to see that $\lambda_0=\beta$, $\phi_0=\widehat{\phi}_0=1$ and then $T_t^{\phi_0}=P_t$.

It is well known that, for any $x \in \mathbb{R}^d$, under Π_x, ξ_t is of Gaussian distribution with mean xe^{-ct} and variance σ_t^2, where $\sigma_t^2 := (1 - e^{-2ct})/(2c)$. The transition density of ξ_t with respect to the probability measure $m(dx)$ on $\mathbb{R}^d$ is given by

$$p(t, x, y) := \left(\frac{1}{2c\sigma_t^2}\right)^{d/2} \exp\left(c\|y\|^2 - \frac{\|y - xe^{-ct}\|^2}{2\sigma_t^2}\right).$$

Note that $p(t, x, x) = (2\pi\sigma_t^2)^{-d/2} \exp\left(-c\frac{1-e^{-ct}}{1+e^{-ct}}\|x\|^2\right)/\varphi(x)$. Thus $a(t) = p(2t, x, x)$ is $L^1(\mathbb{R}^d; m)$-integrable for all $t > 0$ and there is some $t_0 > 0$ so that $a(t) \in L^2(\mathbb{R}^d; m)$-integrable for $t \geq t_0$. Hence Assumption 1.1 holds for ξ.

For any $f \in C_0(\mathbb{R}^d; \mathbb{R})$, we have

$$P_t f(x) = \int_{\mathbb{R}^d} p_t(x, y) f(y) m(dy) = \int_{\mathbb{R}^d} (2\pi)^{-d/2} \exp\left(-\|y\|^2/2\right) f(\sigma_t y + xe^{-ct}) dy.$$

Using the dominated convergence theorem, one can easily check that $P_t f \in C_0(\mathbb{R}^d; \mathbb{R})$. Suppose f is a continuous function with compact support. Let $M_0 > 0$ so that $f(x) = 0$ for $\|x\| \geq M_0$. For any $M > 0$,

$$\begin{aligned}
|P_t f(x) - f(x)| = & \left|\int_{\mathbb{R}^d} (2\pi)^{-d/2} \exp\left(-\|y\|^2/2\right) \left[f(\sigma_t y + xe^{-ct}) - f(x)\right] dy\right| \\
\leq & \int_{\mathbb{R}^d} (2\pi)^{-d/2} \exp\left(-\|y\|^2/2\right) \left|f(\sigma_t y + xe^{-ct}) - f(x)\right| dy \\
\leq & \int_{\|y\| \leq M} (2\pi)^{-d/2} \exp\left(-\|y\|^2/2\right) \left|f(\sigma_t y + xe^{-ct}) - f(x)\right| dy \\
& + 2\|f\|_\infty \int_{\|y\| \geq M} (2\pi)^{-d/2} \exp\left(-\|y\|^2/2\right) dy \\
=: & \ I + II.
\end{aligned}$$

For any $\epsilon > 0$, we choose $M > 0$ such that $II \leq \epsilon/2$. For part I, we claim that, for any $\epsilon > 0$, there exists δ, for $t \leq \delta$,

$$\sup_{\|y\| \leq M} \sup_{x \in \mathbb{R}^d} \left|f(\sigma_t y + xe^{-ct}) - f(x)\right| \leq \epsilon/2.$$

Therefore $I < \epsilon/2$, and then $\|P_t f - f\|_\infty \to 0$ as $t \to 0$.

Now we prove the claim. Note that

$$\left|f(\sigma_t y + xe^{-ct}) - f(x)\right| \leq \left|f(\sigma_t y + xe^{-ct}) - f(xe^{-ct})\right| + \left|f(xe^{-ct}) - f(x)\right|.$$

Since f is uniformly continuous on $\mathbb{R}^d$, there is a constant $\delta_0 > 0$ such that $|f(y) - f(x)| \leq \epsilon/4$ for any x, y satisfying $\|x - y\| \leq \delta_0$. Since $\sigma_t \to 0$ as $t \to 0$, there exists $\delta_1 > 0$ such that, for $t < \delta_1$, $\|\sigma_t\| \leq \delta_0/M$, and then $\sup_{\|y\| \leq M} \sup_{x \in \mathbb{R}^d} |f(\sigma_t y + xe^{-ct}) - f(xe^{-ct})| \leq \epsilon/4$. Choose δ_2, such that for $t \leq \delta_2$, $e^{ct} - 1 \leq \delta_0/M_0$. Then , for $t \leq \delta_2$,

$$|f(xe^{-ct}) - f(x)| \leq |f(xe^{-ct}) - f(x)| \mathbf{1}_{\|x\| \leq M_0 e^{ct}} \leq \epsilon/4,$$

where in the second inequality we use the fact that $\|xe^{-ct}-x\| = \|x\|(1-e^{-ct}) \le M_0(e^{ct}-1) \le \delta_0$. Then, choosing $\delta = \delta_1 \wedge \delta_2$, we prove the claim.

For general $f \in C_0(\mathbb{R}^d;\mathbb{R})$, there exist continuous functions f_n with compact support such that $\|f_n - f\|_\infty \to 0$, as $n \to \infty$. Then

$$\begin{aligned}\|P_t f - f\|_\infty &\le \|P_t f - P_t f_n\|_\infty + \|P_t f_n - f_n\|_\infty + \|f_n - f\|_\infty \\ &\le \|P_t f_n - f_n\|_\infty + 2\|f_n - f\|_\infty.\end{aligned}$$

Letting $t \to 0$ and then $n \to \infty$, we get that $\|P_t f - f\|_\infty \to 0$ as $t \to 0$. Since $T_t^{\phi_0} = P_t$, Assumption 1.3 is satisfied. Therefore for the superprocess in this example, all our assumptions are satisfied.

This example covers Examples 4.1 and 4.6 in [9]. For variable $\alpha(x) = \beta(x)a(x)$, see Example 4.9.

Example 4.2 *[Super outward Ornstein-Uhlenbeck processes]* Let $d \ge 1$, $E = \mathbb{R}^d$. Suppose the spatial motion $\xi = \{\xi_t, \Pi_x\}$ is an OU process on $\mathbb{R}^d$ with infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\sigma^2\Delta + cx\cdot\nabla \text{ on } \mathbb{R}^d,$$

where σ, $c > 0$. Without loss of generality, we assume $\sigma = 1$. Under Π_x, ξ_t is of Gaussian distribution with mean xe^{ct} and variance $(e^{2ct}-1)/(2c)$.

Let $\tilde{\varphi}(x) := (c/\pi)^{-d/2}\, e^{c\|x\|^2}$, and $m(dx) = \tilde{\varphi}(x)dx$. Then ξ is symmetric with respect to the σ-finite measure $m(dx)$. As in the previous example, we suppose that the branching rate function $\beta(x) = \beta$ is a positive constant, and the branching mechanism ψ is given by (4.1). Then for the corresponding superprocess,

$$T_t f(x) = e^{\beta t}\Pi_x\left[f(\xi_t)\right] = e^{\beta t}P_t f(x).$$

The generator of $\{T_t : t \ge 0\}$ is $\mathcal{L} + \beta$.

The transition density of ξ with respect to the measure m is

$$p(t,x,y) = \left(\frac{1}{e^{2ct}-1}\right)^{d/2} \exp\left(-\frac{c}{(1-e^{-2ct})}\left(\|y\|^2 + \|x\|^2 - 2x\cdot ye^{-ct}\right)\right).$$

Thus

$$a_t(x) = p(2t,x,x) = \left(\frac{1}{e^{2ct}-1}\right)^{d/2} \exp\left(-\frac{2c\|x\|^2}{(1+e^{-ct})}\right).$$

It is obvious that $a_t \in L^1(\mathbb{R}^d;m) \cap L^2(\mathbb{R}^d;m)$. Thus Assumption 1.1 is satisfied. Suppose $\beta(x) = \beta \in (cd,\infty)$.

The operator $\mathcal{L} + cd$ is the formal adjoint of the inward OU process with infinitesimal generator $\frac{1}{2}\sigma^2\Delta - cx\cdot\nabla$ on $\mathbb{R}^d$. Since $\varphi(x)$ defined in Example 4.1 is the invariant density of $\frac{1}{2}\sigma^2\Delta - cx\cdot\nabla$

on $\mathbb{R}^d$, $(\mathcal{L}+cd)\varphi = 0$. Thus we have $(\mathcal{L}+\beta)\varphi = (\beta - cd)\varphi$. Since $\varphi \in L^2(\mathbb{R}^d, m)$ and φ is strictly positive everywhere, we know that $\phi_0 = \widehat{\phi}_0 = \varphi$ and $\lambda_0 = \beta - cd$. Thus

$$T_t^{\phi_0} f(x) = \frac{e^{cdt} P_t(f\varphi)(x)}{\varphi(x)} = \tilde{P}_t f(x),$$

where $\tilde{P}_t$ is the semigroup of the inward OU-process with infinitesimal generator

$$\frac{1}{2}\Delta - cx \cdot \nabla \text{ on } \mathbb{R}^d.$$

From the discussion in Example 4.1, we see that Assumption 1.3 is satisfied. Thus, when $\beta(x) = \beta \in (cd, \infty)$, the superprocess of this example satisfies all our assumptions.

This example covers Examples 4.2 in [9].

Example 4.3 Suppose that $\eta = \{\eta_t, \Pi_x\}$ is an m-symmetric Hunt process on E and that η has a transition density $\widetilde{p}(t,x,y)$ with respect to m. Suppose also that $\widetilde{p}$ is strictly positive, continuous and satisfies Assumption 1.1. Let $\{\widetilde{P}_t, t \geq 0\}$ be the transition semigroup of η on $L^2(E; m)$. Since, for each $t > 0$, $\widetilde{P}_t$ is compact, the infinitesimal generator $\widetilde{\mathcal{L}}$ of $\{\widetilde{P}_t, t \geq 0\}$ has discrete spectrum: $0 \geq \widetilde{\lambda}_0 > \widetilde{\lambda}_1 \geq \cdots$. Denote the corresponding normalized eigenfunctions by $\{\widetilde{\phi}_k; k \geq 0\}$, with $\|\widetilde{\phi}_k\|_{L^2(E;m)} = 1$ for every $k \geq 0$. We can choose $\widetilde{\phi}_0$ so that it is strictly positive and continuous. By the spectral representation, we can express $\widetilde{p}(t,x,y)$ by $\sum_{k=0}^{\infty} e^{\widetilde{\lambda}_k t} \widetilde{\phi}_k(x) \widetilde{\phi}_k(y)$. It follows that $\widetilde{p}(t,x,x)$ is decreasing in $t > 0$; see [8, Section 2]. Define

$$\widetilde{P}_t^{\widetilde{\phi}_0} f := e^{-\widetilde{\lambda}_0 t} \frac{\widetilde{P}_t(f\widetilde{\phi}_0)(x)}{\widetilde{\phi}_0(x)}.$$

Assume that $\widetilde{P}_t^{\widetilde{\phi}_0}$ satisfies Assumption 1.3.

Let S_t be a subordinator, independent of ξ, with drift $b > 0$. Then $S_t \geq bt$. Let ϕ be the Laplace exponent of S, that is,

$$\mathbb{E}(e^{-\theta S_t}) = e^{-t\phi(\theta)}, \qquad \theta > 0.$$

Suppose that $\alpha(x) = \alpha$ is a constant function and satisfies $\alpha > \phi(-\widetilde{\lambda}_0)$. We put $\xi_t := \eta_{S_t}$. Let P_t be the semigroup of ξ and $p(t,x,y)$ be the transition density of ξ with respect to m. Then $p(t,x,y) = \mathbb{E}\widetilde{p}(S_t,x,y)$. Since $t \to \widetilde{p}(t,x,x)$ is a decreasing function, $p(2t,x,x) = \mathbb{E}\widetilde{p}(S_{2t},x,x) \leq \widetilde{p}(2bt,x,x)$, which implies that η satisfies Assumption 1.1. Note that $T_t = e^{\alpha t} P_t$, and

$$P_t \widetilde{\phi}_0(x) = \mathbb{E}(\widetilde{P}_{S_t} \widetilde{\phi}_0(x)) = \mathbb{E} e^{\widetilde{\lambda}_0 S_t} \widetilde{\phi}_0(x) = e^{-t\phi(-\widetilde{\lambda}_0)} \widetilde{\phi}_0(x).$$

Thus, $\lambda_0 = \alpha - \phi(-\widetilde{\lambda}_0) > 0$ and $\phi_0 = \widetilde{\phi}_0$. Then

$$T_t^{\phi_0} f(x) = e^{t\phi(-\widetilde{\lambda}_0)} \frac{P_t(f\phi_0)(x)}{\phi_0(x)} = e^{t\phi(-\widetilde{\lambda}_0)} \mathbb{E}\left[\frac{\widetilde{P}_{S_t}(f\phi_0)(x)}{\phi_0(x)}\right].$$

Thus, we have

$$
\begin{aligned}
|T_t^{\phi_0} f(x) - f(x)| &\leq e^{t\phi(-\widetilde{\lambda}_0)} \mathbb{E} \left| \frac{\widetilde{P}_{S_t}(f\phi_0)(x)}{\phi_0(x)} - e^{\widetilde{\lambda}_0 S_t} f(x) \right| \\
&\leq e^{bt\widetilde{\lambda}_0} e^{t\phi(-\widetilde{\lambda}_0)} \mathbb{E} \left[\|\widetilde{P}_{S_t}^{\widetilde{\phi}_0} f - f\|_\infty \right].
\end{aligned}
$$

Since $\|\widetilde{P}_{S_t}^{\widetilde{\phi}_0} f - f\|_\infty \to 0$, as $t \to 0$, and $\|\widetilde{P}_{S_t}^{\widetilde{\phi}_0} f - f\|_\infty \leq 2\|f\|_\infty$, using the dominated convergence theorem, we get that

$$
\lim_{t\to 0} \|T_t^{\phi_0} f - f\|_\infty = 0.
$$

Thus, the superprocess of this example satisfies all our assumptions.

In particular, this example is applicable when η is the outward Ornstein-Uhlenbeck process or inward Ornstein-Uhlenbeck process dealt with in the Examples 4.1 and 4.2.

The next two examples give the cases when α is not a constant function.

Example 4.4 (Pure jump SBM) Suppose that $S = \{S_t, t \geq 0\}$ is a drift-free subordinator. The Laplace exponent ϕ of S can be written in the form

$$
\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\, u(dt), \tag{4.2}
$$

where u is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge t)\, u(dt) < \infty$. The measure u is the Lévy measure of the subordinator (or of ϕ). In this example, we will assume that ϕ is a complete Bernstein function, that is, the measure u has a completely monotone density, which we also denote by u.

Let $W = \{W_t, t \geq 0\}$ be a Brownian motion in $\mathbb{R}^d$ independent of the subordinator S. The subordinate Brownian motion $Y = \{Y_t, t \geq 0\}$ is defined by $Y_t := W_{S_t}$, which is a rotationally symmetric Lévy process with Lévy exponent $\phi(|\xi|^2)$. It is known that the Lévy measure of the process Y has a density given by $x \to j(|x|)$ where

$$
j(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)}\, u(t) dt, \qquad r > 0. \tag{4.3}
$$

Note that the function $r \mapsto j(r)$ is continuous and decreasing on $(0, \infty)$.

Suppose that ϕ satisfies the following growth condition at infinity:

(A): There exist constants $\delta_1, \delta_2 \in (0, 1)$, $a_1 \in (0, 1)$, $a_2 \in (1, \infty)$ and $R_0 > 0$ such that

$$
a_1 \lambda^{\delta_1} \phi(r) \leq \phi(\lambda r) \leq a_2 \lambda^{\delta_2} \phi(r) \qquad \text{for } \lambda \geq 1 \text{ and } r \geq R_0.
$$

See [3] for examples of a large class of symmetric Lévy processes satisfying condition **(A)**.

Suppose D is a bounded $C^{1,1}$ open set with characteristics (R_0, Λ), and let ξ be the subprocess of Y killed upon leaving D. It is known that ξ is a Feller process with strong Feller property in D. Moveover, by [3, Corollary 1.6], ξ has a jointly continuous transition density function $p_D(t, x, y)$ with

respect to the Lebesgue measure on D so that for every $T>0$, there exist $c_1=c_1(R_0,\Lambda,T,d,\phi)\geq 1$ and $c_2=c_2(R_0,\Lambda,T,d,\phi)>0$ such that for $0<t\leq T$, $x,y\in D$,

$$\begin{aligned}
& c_1^{-1}\left(1\wedge\frac{\Phi(\delta_D(x))}{t}\right)^{1/2}\left(1\wedge\frac{\Phi(\delta_D(y))}{t}\right)^{1/2}\Big(\Phi^{-1}(t)^{-d}\wedge tj(|x-y|)\Big)\\
\leq\ & p_D(t,x,y) \qquad (4.4)\\
\leq\ & c_2\left(1\wedge\frac{\Phi(\delta_D(x))}{t}\right)^{1/2}\left(1\wedge\frac{\Phi(\delta_D(y))}{t}\right)^{1/2}\Big(\Phi^{-1}(t)^{-d}\wedge tj(c_2|x-y|/4)\Big).
\end{aligned}$$

Here $\Phi(r):=\frac{1}{\phi(r^{-2})}$, j is the function defined in (4.3), and $\delta_D(x)$ is the Euclidean distance between x and ∂D. Since $p_D(t,x,y)$ is symmetric, $a_t(x)=p_D(2t,x,x)\leq c_2\Phi^{-1}(2t)^{-d}$. Thus, Assumption 1.1 is satisfied.

Suppose that the branching rate function β and the branching mechanism satisfy the assumptions of Subsection 1.2, and that the corresponding superprocess X is supercritical. The corresponding semigroup $\{T_t:t\geq 0\}$ has a continuous density $q(t,x,y)$ satisfying the same two-sided estimates (4.4) with possibly different $c_1\geq 1$ and c_2. Since $\phi_0(x)=e^{\lambda_0 t}T_t\phi_0(x)$, by (4.4),

$$\phi_0(x)\asymp\Phi(\delta_D(x))^{1/2}.$$

We now show that Assumption 1.3 holds. Suppose $f\in C_0(D)$. For any given $\varepsilon>0$, there $\delta>0$ so that $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$. Hence by the display above and (4.4), for small $t>0$,

$$\begin{aligned}
& \sup_{x\in D}|T_t^{\phi_0}f(x)-f(x)|\\
=\ & \sup_{x\in D}\frac{e^{-\lambda_0 t}\left|\Pi_x\left[e^{\int_0^t\alpha(\xi_s)\,du}\phi_0(\xi_t)\left(f(\xi_t)-f(\xi_0)\right)\right]\right|}{\phi_0(x)}\\
\leq\ & \varepsilon+\sup_{x\in D}\frac{e^{-\lambda_0 t}\left|\Pi_x\left[e^{\int_0^t\alpha(\xi_s)\,du}\phi_0(\xi_t)\left|f(\xi_t)-f(\xi_0)\right|;|\xi_t-\xi_0|\geq\delta\right]\right|}{\phi_0(x)}\\
\leq\ & \varepsilon+\sup_{x\in D}c\frac{e^{(-\lambda_0+\|\alpha\|_\infty)t}\|\phi_0\|_\infty\|f\|_\infty\Pi_x(|\xi_t-\xi_0|\geq\delta)}{\phi_0(x)}\\
\leq\ & \varepsilon+\sup_{x\in D}c\frac{\Phi(\delta_D(x))^{1/2}t^{-1/2}\int_{y\in D:|y-x|>\delta}tj(c_2|y-x|/4)dy}{\Phi(\delta_D(x))^{1/2}}\\
\leq\ & \varepsilon+c\sqrt{t}\int_{|z|\geq c_2\delta/4}(1\wedge|z|^2)j(|z|)dz. \qquad (4.5)
\end{aligned}$$

It follows that $\lim_{t\to 0}\|T_t^{\phi_0}f-f\|_\infty=0$ and Assumption 1.3 is satisfied.

Example 4.5 (SBM with Gaussian component) Suppose that $S=\{S_t,t\geq 0\}$ is a subordinator with drift $b>0$. The Laplace exponent ϕ of S can be written in the form

$$\phi(\lambda)=b\lambda+\int_0^\infty(1-e^{-\lambda t})\,u(dt), \qquad (4.6)$$

where u is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge t)\, u(dt) < \infty$. Without loss of generality we assume that $b = 1$. In this example, we will assume that ϕ is a complete Bernstein function and that the Lévy density $u(t)$ of S satisfies the following growth condition on $u(t)$ in (4.2) near zero: For any $M > 0$, there exists $c = c(M) > 1$ such that

$$u(r) \le c u(2r), \quad r \in (0, M). \tag{4.7}$$

Let $W = \{W_t, t \ge 0\}$ be a Brownian motion in $\mathbb{R}^d$ independent of the subordinator S. The subordinate Brownian motion $Y = \{Y_t, t \ge 0\}$ is defined by $Y_t := W_{S_t}$, which is a rotationally symmetric Lévy process with Lévy exponent $\phi(|\xi|^2)$. It is known that the Lévy measure of the process Y has a density $j(|x|)$ given by (4.3).

For any open set $D \subset \mathbb{R}^d$ and positive constants c_1 and c_2, we define

$$\begin{aligned} & h_{D,c_1,c_2}(t,x,y) \\ := & \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right)\left(t^{-d/2} e^{-c_1|x-y|^2/t} + t^{-d/2} \wedge (tj(c_2|x-y|))\right). \end{aligned} \tag{4.8}$$

Suppose D is a bounded $C^{1,1}$ open set with characteristics (R_0, Λ), and let ξ be the subprocess of Y killed upon leaving D. It is known that ξ is a Hunt process symmetric with respect to the Lebesgue measure on D and that ξ has a strictly positive continuous transition density $p_D(t,x,y)$ with respect to the Lebesgue measure on D. We assume the following upper bound condition on the transition density function $\widetilde{p}(t,|x|)$ of Y: for any $T > 0$, there exist $C_j \ge 1$, $j = 1,2,3$, such that for all $(t,r) \in (0,T] \times [0, \mathrm{diam}(D)]$,

$$\widetilde{p}(t,r) \le C_1 \left(t^{-d/2} e^{-r^2/C_2 t} + t^{-d/2} \wedge (tj(r/C_3))\right). \tag{4.9}$$

It is established in [6] that the above estimate holds for a large class of symmetric diffusion processes with jumps with $D = \mathbb{R}^d$. Using Meyer's method of removing and adding jumps, it can be shown that (4.9) is true for a larger class of symmetric Markov processes, including subordinate Brownian motions with Gaussian components under some additional condition. See the paragraph containing (1.12) in [4] for more information.

The following is proved in [4, Theorem 1].

(i) For every $T > 0$, there exist $c_1 = c_1(R_0, \Lambda_0, \lambda_0, T, \psi, d) > 0$ and $c_2 = c_2(R_0, \Lambda_0, \lambda_0, d) > 0$ such that for all $(t,x,y) \in (0,T] \times D \times D$,

$$p_D(t,x,y) \;\ge\; c_1\, h_{D,c_2,1}(t,x,y). \tag{4.10}$$

(ii) If D satisfies (4.9), then for every $T > 0$, there exists $c_3 = c_3(R_0, \Lambda_0, T, d, \psi, C_1, C_2, C_3, d) > 1$ such that for all $(t,x,y) \in (0,T] \times D \times D$,

$$p_D(t,x,y) \;\le\; c_3\, h_{D,C_4,C_5}(t,x,y), \tag{4.11}$$

where $C_4 = (16C_2)^{-1}$ and $C_5 = (8 \vee 4C_3)^{-1}$.

Let $E = D$ and m be the Lebesgue measure on D. Since $p_D(t,x,y)$ is symmetric, $a_t(x) = p_D(2t,x,x) \le ct^{-d/2}$. Thus, Assumption 1.1 is satisfied.

Suppose that the branching rate function β and the branching mechanism satisfy the assumptions of Subsection 1.2, and that the corresponding superprocess X is supercritical. Using the above two-sided heat kernel estimate for ξ, we can establish in a similar way as in Example 4.4 that Assumption 1.3 also holds.

Remark 4.6 In fact, in the two examples above, ξ does not need to be a subordinate Brownian motion killed upon leaving D. All we need are the heat kernel estimates like (4.4) or (4.10)-(4.11). For example, suppose Y^D is the subprocess of some subordinate Brownian motion Y killed upon leave D that has the property (4.4) or (4.10)-(4.11). Let ξ be a Markov process obtained from Y^D though a Feynman-Kac transform with bounded potential function. Then ξ enjoys the property (4.4) or (4.10)-(4.11). For other examples of processes that satisfy two-sided bounds similar to (4.4), including censored stable processes in $C^{1,1}$ open sets and their local and non-local Feynman-Kac transforms, see [2]. Our main results are applicable to these processes as well.

In all the examples above, the spatial motion ξ is symmetric. Now we give two examples where the spatial motion ξ is not symmetric.

Example 4.7 Suppose $d \ge 3$ and that $\nu = (\nu^1, \cdots, \nu^d)$, where each ν^j is a signed measure on $\mathbb{R}^d$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{B(x,r)} \frac{|\nu^j|(dy)}{|x-y|^{d-1}} = 0.$$

Let $\xi^{(1)} = \{\xi^{(1)}_t, t \ge 0\}$ be a Brownian motion with drift ν in $\mathbb{R}^d$, see [1]. Suppose that D is a bounded domain in $\mathbb{R}^d$. Let $M > 0$ so that $B(0, M/2) \supset D$. Put $B = B(0, M)$. Let G_B be the Green function of $\xi^{(1)}$ in B and define $H(x) := \int_B G_B(y,x)dy$. Then H is a strictly positive continuous function on B. Let ξ be the process obtained by killing $\xi^{(1)}$ upon exiting D. ξ is a Hunt process and it has a strictly positive continuous transition density $\widetilde{p}(t,x,y)$ with respect to the Lebesgue measure on D. Let $E = D$ and m be the measure defined by $m(dx) = H(x)dx$. It follows from [15, 16] that ξ has a dual process with respect to m. The transition density of ξ with respect to m is given by $p(t,x,y) = \widetilde{p}(t,x,y)/H(y)$.

Suppose further that D is $C^{1,1}$, then it follows from [14, Theorem 4.6] that there exist $c_1 > 1, c_2 > c_3 > 0$ such that for all $(t,x,y) \in (0,1] \times D \times D$,

$$\begin{aligned} & c_1^{-1} t^{-d/2} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) \exp\left(-\frac{c_2|x-y|^2}{t}\right) \\ \le \quad & \tilde{p}(t,x,y) \le c_1 t^{-d/2} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) \exp\left(-\frac{c_3|x-y|^2}{t}\right). \end{aligned}$$

It follows from the display above and the semigroup property that, for any $t > 0$, $\tilde{p}(t,x,y)$ is bounded. By [16, (2.6)], $H(x) \asymp \delta_B(x)$. So for $x \in D$, $c \le H(x) \le C$, where $c, C > 0$. Thus, $p(t,x,y)$ is also bounded in D and m is a finite measure. Thus Assumption 1.1 is satisfied.

Suppose that the branching rate function β and the branching mechanism satisfy the assumptions of Subsection 1.2, and that the corresponding superprocess X is supercritical. Using the above two-sided heat kernel estimate for ξ, we can establish in a similar way as in Example 4.4 that Assumption 1.3 also holds.

Example 4.8 Suppose $d \geq 2$, $\alpha \in (1,2)$, and that $\nu = (\nu^1, \cdots, \nu^d)$, where each ν^j is a signed measure on $\mathbb{R}^d$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{B(x,r)} \frac{|\nu^j|(dy)}{|x-y|^{d-\alpha+1}} = 0.$$

Let $\xi^{(2)} = \{\xi_t^{(2)}, t \geq 0\}$ be an α-stable process with drift ν in $\mathbb{R}^d$, see [17]. Suppose that D is a bounded open set in $\mathbb{R}^d$ and suppose $M > 0$ is such that $D \subset B(0, M/2)$. Put $B = B(0, M)$. Let G_B be the Green function of $\xi^{(2)}$ in B and define $H(x) := \int_B G_B(y,x)dy$. Then H is a strictly positive continuous function on B. Let ξ be the process obtained by killing $\xi^{(2)}$ upon exiting D. ξ is a Hunt process and it has a strictly positive continuous transition density $\widetilde{p}(t,x,y)$ with respect to the Lebesgue measure on D. Let $E = D$ and m be the measure defined by $m(dx) = H(x)dx$. It follows from [5, Section 5] and [17] that ξ has a dual process with respect to m. The transition density of ξ with respect to m is given by $p(t,x,y) = \widetilde{p}(t,x,y)/H(y)$. By [5, Corollary 1.4] and [18], we can check that $H(x) \asymp \delta_B(x)^{\alpha/2}$. Thus, for $x \in D$, $c \leq H(x) \leq C$, for some $c, C > 0$.

Suppose further that D is $C^{1,1}$, then it follows from [5, Theorem 1.3] and [18] that there exists $c_1 > 1$ such that for all $(t,x,y) \in (0,1] \times D \times D$,

$$\begin{aligned} & c_1^{-1}\left(1 \wedge \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D^{\alpha/2}(y)}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \\ \leq\ & \tilde{p}(t,x,y) \leq c_1\left(1 \wedge \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D^{\alpha/2}(y)}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \end{aligned}$$

It follows from the display above and the semigroup property that, for any $t > 0$, $\tilde{p}(t,x,y)$ is bounded. Since H is bounded between two positive constants, Assumption 1.1 is satisfied.

Suppose that the branching rate function β and the branching mechanism satisfy the assumptions of Subsection 1.2, and that the corresponding superprocess X is supercritical. Using the above two-sided heat kernel estimate for ξ, we can establish in a similar way as in Example 4.4 that Assumption 1.3 also holds.

In the following example, our main result does not apply directly. However, we could apply our main result after a transform.

Example 4.9 Suppose the spatial motion $\xi = \{\xi_t, \Pi_x\}$ is an OU-process on $\mathbb{R}^d$ with infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\sigma^2\Delta - cx\cdot\nabla \text{ on } \mathbb{R}^d,$$

where $\sigma,\ c>0$. Without loss of generality, we assume $\sigma=1$. Let $\varphi(x):=(c/\pi)^{d/2}\,e^{-c|x|^2}$, and $m(dx)=\varphi(x)dx$. Then (ξ,Π_x) is symmetric with respect to the probability measure $m(dx)$.

Let $a(x)=c_1|x|^2+c_2$ with $c_1,c_2>0$, and let P^a_t be the Feynman-Kac semigroup,

$$P^a_t f(x):=\Pi_x\left[e^{\int_0^t a(\xi_s)ds}f(\xi_t)\right].$$

Suppose $c>\sqrt{2c_1}$ and write $\upsilon=\frac{1}{2}(c-\sqrt{c^2-2c_1})$. Let

$$\lambda_c:=\inf\{\lambda\in\mathbb{R}:\ \text{there exists } u>0 \text{ such that } (\mathcal{L}+a-\lambda)u=0 \text{ in } \mathbb{R}^d\}$$

be the generalized principal eigenvalue. Let h denote the corresponding ground state, i.e., $h>0$ such that $(\mathcal{L}+a-\lambda_c)h=0$. As is indicated in [10], $\lambda_c=c_2+d\upsilon>0$ and $h(x)=\left(\frac{c-2\upsilon}{c}\right)^{d/2}\exp\{\upsilon\|x\|^2\}$.

Note that $h=e^{-\lambda_c t}P^a_t h$ on $\mathbb{R}^d$. Let Π^h_x be defined as in (1.15) with ϕ_0 replaced by h. The transformed process (ξ,Π^h_x) is also an OU-process with infinitesimal generator $\frac{1}{2}\Delta-(c-2\upsilon)x\cdot\nabla$ on $\mathbb{R}^d$.

Let $\psi(x,z)=-a(x)z+\alpha(x)z^2$, where $\alpha\in C^\eta(\mathbb{R}^d)$, $\alpha(x)>0$ for all $x\in\mathbb{R}^d$. A superprocess X with spacial motion ξ, branching rate $\beta(x)=1$ and branching mechanism ψ can be defined by $X=\frac{1}{h}X^h$, where X^h is the superprocess with spacial motion (ξ,Π^h_x), branching rate $\beta(x)=1$ and branching mechanism $\psi^h(x,z)=-\lambda_c z+h(x)\alpha(x)z^2$.

Assume that $h\alpha$ is bounded in $\mathbb{R}^d$. Then, for X^h, we have $m^h(dx)=(\frac{c-2\upsilon}{\pi})^{d/2}e^{-(c-2\upsilon)|x|^2}\,dx$, $\lambda^h_0=\lambda_c$ and $\phi^h_0=1$. From the discussion in Example 4.1, we see that the Assumption 1.1 and Assumption 1.3 are satisfied for the superprocess X^h. Then, there exists $\Omega_0\subset\Omega$ of probability one (that is, $\mathbb{P}_\mu(\Omega_0)=1$ for every $\mu\in\mathcal{M}_F(\mathbb{R}^d)$) such that, for every $\omega\in\Omega_0$ and for every bounded Borel measurable function $f\ge 0$ on $\mathbb{R}^d$ with $f/h\le c$ for some $c>0$ and that the set of discontinuous points of f has zero m-measure, we have

$$\begin{aligned}\lim_{t\to\infty}e^{-\lambda^h_0 t}\langle f/h,X^h_t\rangle(\omega) &= W_\infty(\omega)\int_{\mathbb{R}^d}(f/h)(y)m^h(dy)\\ &= W_\infty(\omega)\Big(\frac{c}{\pi}\Big)^{d/2}\int_{\mathbb{R}^d}f(y)e^{(\upsilon-c)|y|^2}\,dy,\end{aligned}\tag{4.12}$$

where $W_\infty(\omega)$ is the limit of the martingale $W_t:=e^{-\lambda_c t}\langle 1,X^h_t\rangle=e^{-\lambda_c t}\langle h,X_t\rangle$ as $t\to\infty$. We rewrite (4.12) to get the limit result on X:

$$\begin{aligned}\lim_{t\to\infty}e^{-\lambda_c t}\langle f,X_t\rangle(\omega) &= W_\infty(\omega)\int_{\mathbb{R}^d}\left(\frac{c}{\pi}\right)^{d/2}e^{(\upsilon-c)\|y\|^2}f(y)dy\\ &= W_\infty(\omega)\int_{\mathbb{R}^d}\widetilde{\phi}_0(y)f(y)dy,\end{aligned}\tag{4.13}$$

where $\widetilde{\phi}_0=\left(\frac{c}{\pi}\right)^{d/2}e^{(\upsilon-c)\|y\|^2}$. Since h is bounded from below, in the weak topology, $e^{-\lambda_c t}X_t\to W_\infty(\omega)\widetilde{\phi}_0(x)dx$, $\mathbb{P}_\mu$-a.s., for any $\mu\in\mathcal{M}_F(\mathbb{R}^d)$. This example covers [9, Example 4.7].

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